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THREE-DIMENSIONAL PLASTIC FLOW UNDER UNIFORM STRESS

by

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Abstract. In view of the role that regions of uniform stress play in the theory of plane plastic flow, the most general velocity field is investigated that is possible under a uniform state of stress at the yield limit. Two cases are distinguished according to whether all principal components of the stress deviation are different from zero ("regular case") or not ("degenerate case"). The analytical description of the velocity field involves two arbitrary functions in the degenerate case but no such functions in the regular case.

1. Introduction. The theory of the slip line field in plane plastic flow is one of the best developed branches of the mathematical theory of perfectly plastic solids. Among the problems that have been solved in this field, those involving rectilinear boundaries are predominant. In many of these problems, the slip line field contains finite regions of uniform stress. The most general velocity field possible in such a region is obtained by the superposition of two arbitrary shear flows in the directions of the maximum shearing stresses. The analytical description of this velocity field involves two arbitrary functions of one variable each.

Whereas fields of plane plastic flow have been studied systematically (see, for instance, Ref. 1, Chaps. 6-9, or Ref. 2, Chaps. 5-7), three-dimensional plastic flow is practically unexplored. Finzi (Ref. 3) and Thomas (Ref. 4) obtained some general

results, but did not construct examples of three-dimensional flows. De Simoni (Ref. 5) attempted to construct a three-dimensional plastic flow field, but closer inspection of his example reveals this to be a field of plane flow referred to coordinate axes that are oblique to the plane of flow. A non-trivial and genuinely three-dimensional field of plastic flow is found in a paper by Hill (Ref. 6). This flow field describes the incipient plastic flow in a prismatic bar made of a plastic rigid material and subjected to combined tension, torsion, and bending. The equations for the velocity components (Ref. 6, Eqs. 17) contain an unknown function, however, and the non-linear differential equation for this function has not yet been solved. When this function is set equal to zero and the necessary adjustment of constants is made, Hill's flow field reduces to a special case of the "regular" field discussed in the present paper.

In view of the role that regions of uniform stress play in the theory of plane plastic flow, it seems worthwhile to investigate the most general three-dimensional flow field that is possible under a uniform state of stress at the yield limit. It will be seen that, contrary to what is the case for plane flow, the analytical description of the most general three-dimensional flow field possible under uniform stress does not, in general, involve any arbitrary functions.

2. *Fundamental equations.* Choose the rectangular axes X, Y, and Z, in the principal directions of the uniform stress field and denote the given principal components of the stress deviation by K, L, and M, and the unknown velocity components by

U, V, and W. According to Mises' theory of plasticity (Ref. 7),
the components of the velocity strain at a generic point are
proportional to the components of the stress deviation at this
point, the non-negative factor of proportionality being a func-
tion of position. Applied to the present case, this flow rule
leads to the following equations in which subscripts denote dif-
ferentiation with respect to the coordinates:

$$U_X = \lambda K, \qquad V_Y = \lambda L, \qquad W_Z = \lambda M; \qquad (1)$$

$$U_Y + V_X = 0, \qquad V_Z + W_Y = 0, \qquad W_X + U_Z = 0. \qquad (2)$$

In these equations, λ denotes the non-negative factor of propor-
tionality.

Since K, L, and M, are constants, elimination of the
velocity components between Eqs. (1) and (2) leads to the follow-
ing differential equations for the unknown function λ:

$$K\lambda_{YY} + L\lambda_{XX} = 0, \qquad L\lambda_{ZZ} + M\lambda_{YY} = 0, \qquad M\lambda_{XX} + K\lambda_{ZZ} = 0; (3)$$

$$K\lambda_{YZ} = 0, \qquad\qquad L\lambda_{ZX} = 0, \qquad\qquad M\lambda_{XY} = 0. (4)$$

These are the equations of compatibility (see, for instance, Ref.
8, p. 27, Eqs. 10.10) for the components of the velocity strain
when these are expressed in terms of the components of the stress
deviation and the factor λ.

In discussing Eqs. (3) and (4), we must keep in mind
that K, L, and M, are the principal components of the stress
deviation. Accordingly,

$$K + L + M = 0. \qquad (5)$$

The following two cases must be discussed separately.

1) None of the quantities K, L, and M, vanishes. Equations (3) then yield

$$\lambda_{XX} = \lambda_{YY} = \lambda_{ZZ} = 0. \tag{6}$$

This will be called the regular case.

2) One of the quantities K, L, and M, vanishes. If, for instance, M = 0, we have L = - K, by (5), and Eqs. (3) and (4) furnish

$$\lambda_{XX} - \lambda_{YY} = 0, \quad \lambda_{ZZ} = 0; \tag{7}$$

$$\lambda_{YZ} = 0, \quad \lambda_{ZX} = 0. \tag{8}$$

This will be called the degenerate case.

For the regular case, Eqs. (4) and (5) yield

$$\lambda = AX + BY + CZ + D, \tag{9}$$

where A, B, C, and D, are arbitrary constants. For the singular case, it follows from Eqs. (7) and (8) that λ has the form

$$\lambda = f(X + Y) + g(X - Y) + CZ, \tag{10}$$

where C again denotes an arbitrary constant.

3. Regular velocity fields. With λ as given by Eq. (9), integration of Eqs. (1) and (2) will yield a velocity field that is compatible with the considered uniform stress field. Since λ is non-negative, however, such a velocity field can be constructed only in the half-space in which the right-hand side of (9) is

non-negative. The following analysis is simplified by the intro-
duction of new rectangular coordinates x, y, and z, that are
chosen so that this half-space corresponds to $z \geq 0$.

Let k, ℓ, and m, and p, q, and r, be the normal and
shear components of the stress deviation with respect to the new
coordinate axes, and u, v, and w, the components of the velocity.
Equations (1) and (2) must now be replaced by

$$u_x = \lambda k, \qquad v_y = \lambda \ell, \qquad w_z = \lambda m; \qquad (11)$$

$$u_y + v_x = 2\lambda p, \quad v_z + w_y = 2\lambda q, \quad w_x + u_z = 2\lambda r, \qquad (12)$$

where
$$\lambda = cz, \quad (c > 0). \qquad (13)$$

To within a velocity field that corresponds to a rigid
body motion, the most general velocity field compatible with
thes. equations is given by

$$\left. \begin{aligned} u &= c(kxz + pyz + rz^2), \\ v &= c(pxz + \ell yz + qz^2) \\ w &= -\tfrac{1}{2} c(kx^2 + \ell y^2 + 2pxy - mz^2). \end{aligned} \right\} \quad (z > 0) \qquad (14)$$

This velocity field involves no arbitrary functions, in fact no
arbitrary constants other than c.

For given values of the components of the stress devia-
tion, the velocity field (14) is valid only in $z \geq 0$. Under cer-
tain conditions, it is possible, however, to continue this field
into z < 0 by admitting different uniform states of stress on the
two sides of the plane z = 0. We use the prime to distinguish

quantities in $z < 0$ from the corresponding quantities in $z > 0$.
Since λ cannot be negative, the constant c' must be negative. If
the velocity field (14) is to be valid throughout space, we must
therefore set $k' = - k$, $\ell' = - \ell$, ... , $r' = - r$. Equilibrium
between the tractions transmitted across $z = 0$ requires, however,
that $q' = q$, $r' = r$. There is no such condition for m and m'
because the normal stresses in the z direction are obtained by
adding, to m and m', the respective mean normal stresses which
do not appear elsewhere in our analysis. It follows from the
preceding discussion that

$$q = q' = r = r' = 0. \tag{15}$$

With this restriction, the velocity field (14) can be continued
into $z < 0$. The plane $z = 0$ is then a discontinuity surface of
the stress field (for a discussion of such discontinuity surfaces
see Ref. 9). The application of this type of stress and velocity
fields to a plastic plate under uniform bending and twisting
moments is obvious.

4. **Degenerate velocity fields.** With λ as given by
Eq. (10), integration of Eqs. (1) and (2) yields the following
result: to within a velocity field that corresponds to a rigid
body motion, the most general velocity field possible in the de-
generate case is given by

$$U = K [F(X + Y) + G(X - Y) + CXZ],$$

$$V = -K[F(X + Y) - G(X - Y) + CYZ], \tag{16}$$

$$W = - \tfrac{1}{2} CK(X^2 - Y^2),$$

where the functions F and G have as derivatives the functions f
and g appearing in (10). This velocity field is, of course,
restricted to the region in which λ as given by Eq. (10) is non-
negative.

The known results for plane plastic flow under constant
stress are obtained from (16) by setting C = 0. The general
degenerate field (16) results from the superposition of the field
of plane plastic flow represented by the functions F and G, and
the field represented by the terms with C in Eqs. (16). The
velocities of the latter field tend to tilt the planes X = const.
or Y = const. about their intersections with the plane Z = 0 and
to transform the planes Z = const. into congruent hyperbolic
paraboloids.

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